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Weak Convergence in Hilbert Space and Weak Uncertainty Relations in Quantum Theory

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Abstract

A suitable weak topology is considered on the Hilbert phase space of a quantum-mechanical system. It is then shown that if two bounded observables of the system have no common eigenvector, the sum of their variances in any state is always greater than some positive constant. Consequences of this result on some observables of simple physical systems are examined. First of all, the case of the position and momentum of the elementary particle in one dimension is studied and a comparation with Heisenberg's indeterminacy principle is carried out. Then, the case of angular variables is also examined, with special emphasis on spin 1/2. An experiment with neutrons is finally suggested and analysed with the help of the theory developed.

1. *Introduction*

Given two observables X, Y canonically related—i.e. obeying the commutation rule $[X, Y] = i\hbar$—Heisenberg's uncertainty principle asserts that the product of their variances is always greater than or equal to $(h/2)^2$. Our aim, in the present paper, is to deal with a related sort of 'weak uncertainty principle': (WUP) if two bounded observables X and Y have no common eigenstate, then the sum of their variances is always greater than some positive constant. This is proved as the theorem 3.1, at the beginning of Section 3. Some preliminary mathematical tools, needed for the proof and essentially of a topological nature, are developed in Section 2.

As it can be verified from the considerations which will follow, WUP seems worthy of attention in its own. Indeed, it can be applied not only to

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pairs of observables canonically related† but also to observables which *are not* canonically related—thus, to which Heisenberg's relation does not apply.‡ Furthermore, it may eventually lead to quite a variety of technical results—an example of which is found in Alvim (1970).

But there is still other reasons for considering WUP. Indeed, when studying the lattice-theoretical foundations of (non-Relativistic) Quantum Mechanics,§ two species of pairs of observables are naturally singled out. One species is formed by the pairs whose components do commute and the other species by those whose components have no common eigenstates. Now, WUP applies exactly to this last kind of pairs. The importance of this fact is better understood if we heed von Neumann's comments on Bohr's 1927 'complementarity lecture' (as described in Jammer (1966), p. 354).

Furthermore, WUP is a purely qualitative assertion, in the sense that its premise depends only on a 'non-numerical' property of X and Y. As a consequence of such a property, it follows the existence of a lower bound for the sum of the variances of X and Y. This situation must be compared with the orthodox quantum mechanical context, where Heisenberg's uncertainty relation is deduced from a quantitative law (i.e. $[X, Y] = i\hbar$). Heisenberg saw in his indeterminacy principle the 'direct intuitive interpretation' of the canonical commutation rules (cf. Jammer (1966), p. 328). On the other hand, as noted by Born,¶ the canonical commutation rules are the equations which introduce Planck's constant in Quantum Mechanics. From all this and assuming a quite unorthodox point of view, we might consider WUP as a step—yet rather primitive—towards introducing $\hbar$ in (non-Relativistic) Quantum Mechanics through *intuitive*, *elementary* and *qualitative* principles.

Finally, we remark that very often we will call 'the orthodox context of Quantum Mechanics' today's Quantum-Mechanical formalism *without* WUP. It must be emphasised, however, that WUP is deduced entirely within the orthodox Hilbert-space formalism of (non-Relativistic) Quantum-Mechanics. Furthermore, it would still be valid even if some parts of such a formalism fail to hold. Thus, in Sections 3 to 5 we contemplate the possibility that some operators, which in the orthodox context are supposed to obey the canonical commutation rules, actually may not do so‖ but still obey WUP. This supposition poses itself as a natural object of study, due to the cleavage of the pairs of 'lattice-theoretical observables' into the two species we have indicated above.

† Cf. Section 3; in Section 4 we examine a closely related situation.

‡ Cf. Section 5.

§ We must add here: 'such a study being carried out from the standpoint of an *atomistic-operational* heuristics'. The operational standpoint was developed in Alvim (1969).

¶ Cf. Born *et al.* (1926), ch. 1, comments on equation (5). See also Jammer (1966), p. 211.

‖ Or, obey the canonical commutation rules but have the variances in their measurements acted upon by some kind of yet unknown interference.

2. *Weak Convergence on the Set of States of a Quantum-Mechanical System*

Let H be a complex separable Hilbert space and B the real vector space of bounded linear self-adjoint operators on H. Let S be the set of all linear functionals $p: B \to R$ which are of the form

$$p(X) = \sum_{j=1}^{\infty} \lambda_j \langle Xv_j, v_j \rangle$$

where the v_j are normalised elements of H and the λ_j real coefficients such that $0 \leqq \lambda_j \leqq 1$, $\sum_{j=1}^{\infty} \lambda_j = 1$.

Definition 2.1—By *the weak topology of B* we will understand the weakest topology on B under which the linear functionals $p \in S$ are continuous. The space B, endowed with this topology, is then a topological vector space which will be denoted henceforward by wB.

We are specially interested in the dual $\overline{wB}$ of wB; that is to say, in the real vector space of continuous linear mappings from wB into R. By definition 2.1 we have $S \subset \overline{wB}$.

Definition 2.2—By *the weak topology of* $\overline{wB}$ we will understand the weakest topology on $\overline{wB}$ under which the linear functionals $\cdot(X): f \to f(X)$, from $\overline{wB}$ into R, are continuous for every $X \in B$. The space $\overline{wB}$, endowed with this topology, is then a topological vector space which will be denoted henceforward by $w\overline{wB}$.

The weak topology of $\overline{wB}$ is Hausdorff and locally convex; a basis of neighbourhoods for $f_0 \in w\overline{wB}$ is given by the sets $V(f_0, N, \delta) = \{f: f \in w\overline{wB}, |f(X) - f_0(X)| < \delta$ for all $X \in N\}$—where N runs over all the finite sets of elements of wB and δ runs over the positive reals. A sequence $\{f_j\} \subset w\overline{wB}$ *converges weakly* (*to the element* $f \in w\overline{wB}$) if and only if $\lim f_j(X) = f(X)$ for every $X \in wB$.

Now, let us recall that if $X \in B$, *the norm of X* is defined as

$$\|X\| = \sup\{|\langle Xv, Xv \rangle^{1/2}| : v \in H, \langle v, v \rangle = 1\}$$

By its turn, for $f \in \overline{wB}$ *the norm of f* is defined as

$$\|f\| = \sup\{|f(X)| : X \in B, \|X\| = 1\}$$

We have a result similar to Alaoglu's theorem:

Theorem 2.3—The unity ball $\odot = \{f: f \in w\overline{wB}, \|f\| \leqq 1\}$ of $w\overline{wB}$ is compact.

Proof—See, for instance, Dunford & Schwartz (1958), v. 4.1–v. 4.2.

But besides being compact when endowed with its relative topology as a subset of $w\overline{wB}$, the unity ball is then also a metrisable topological space. In order to show this, we need some auxiliary notation and results.

Let A be the complex vector space of all linear bounded transformations in H and C the subspace of A formed by all the compact linear transformations in H. By wA we understand A endowed with the weakest topology under which the linear functionals $X \to \sum_{j=1}^{\infty} \langle Xv_j, w_j \rangle$ from A into $\mathbb{C}$—where $\{v_j\}$, $\{w_j\}$ are any sequences of elements of H such that $\sum_{j=1}^{\infty} (\langle v_j, v_j \rangle + \langle w_j, w_j \rangle) < \infty$—are all continuous.† The dual of wA will be denoted $\overline{wA}$ (we will not need to define any topology on it; $\overline{wA}$ will be considered merely as a complex vector space).

The norm $\|X\|$ of an element $X \in A$ being defined in the usual way, we denote by nC the space C endowed with the uniform operator topology. In the dual $\overline{nC}$ of nC we consider also the usual norm topology—the resulting topological vector space being denoted by $n\overline{nC}$ and its dual by $\overline{n\overline{nC}}$.

It is known (see, for instance, Dixmier (1950), p. 394) that there exists an isometric isomorphism $\theta: \overline{n\overline{nC}} \to A$ such that, if Ξ is the canonical imbedding of nC into $\overline{n\overline{nC}}$, then $\theta(\Xi(X)) = X$, $\forall X \in nC$. Any $f \in \overline{nC}$ may be taken as the restriction to C of the linear functional $\bar{f}$ over A whose action is given by the formula $\bar{f}(X) = \theta^{-1}(X).(f)$, $\forall X \in A$. It happens furthermore that the elements of $\overline{wA}$ are exactly the 'extensions' $\bar{f}$ of the functionals $f \in \overline{nC}$, constructed as just described (Dixmier (1950), p. 398).

We also know‡ that nC contains a dense countable subset $\{U_j\}$. Let us suppose now that $\bar{f}_0(U_j) = 0$ for all U_j and some $\bar{f}_0 \in \overline{wA}$. But $\bar{f}_0(U_j) = 0$ implies $\bar{f}_0(U_j) = \theta^{-1}(U_j).(f_0) = 0$ and as $\theta(\Xi(U_j)) = U_j$, it follows that $\Xi(U_j).(f_0) = f_0(U_j) = 0$ for all U_j. From this we can deduce that f_0 is then the null element of $\overline{nC}$ and, going back to $\bar{f}_0(X) = \theta^{-1}(X).(f_0)$, we finally conclude that $\bar{f}_0$ is also the null element of $\overline{wA}$.

Taking into account the above result, we proceed to prove the

Lemma 2.4—There is a countable set $\{X_j\} \subset wB$ such that if $f \in \overline{wB}$ and $f(X_j) = 0$ for all X_j, then f is the null functional of $\overline{wB}$.

Proof—From the above considerations, it follows that there exists a countable set $\{U_j\} \subset wA$ such that if $f_0 \in \overline{wA}$ and $f_0(U_j) = 0$ for all U_j, then f_0 is the null functional of $\overline{wA}$. Let us write each $X \in wA$ with the help of self-adjoint transformations, in the usual way: $X = X_1 + iX_2$ with X_1, $X_2 \in wB$. Now suppose that $f \in \overline{wB}$ and $f(U_{1j}) = f(U_{2j}) = 0$ for all the self-adjoint components U_{1j}, U_{2j} of the transformations U_j. But the functional $\bar{f}$ defined by the equation $\bar{f}(X) = f(X_1) + if(X_2)$ belongs to

† This topology corresponds to the 'topologie ultrafaible' defined in Dixmier (1953) and to the 'topologie $\sigma(\mathscr{B},\mathscr{T}')$' of Dixmier (1950).

‡ Dixmier (1950), p. 392, proposition 4.

$\overline{wA}$.† Hence, as $\bar{f}(U_j) = f(U_{1j}) + if(U_{2j}) = 0 + 0 = 0$ for all j, $\bar{f}$ is the null element of $\overline{wA}$—and its restriction f to wB turns out to be the null element of $\overline{wB}$.

Lemma 2.5—Let $\{X_j\}$ be the same set described in the previous lemma and $\dot{X}_j = X_j/\|X_j\|$. Then,

$$d(f,g) = \sum_{j=1}^{\infty} \frac{|(f-g).\dot{X}_j|}{2^j\,(1 + |(f-g)\,\dot{X}_j|)}$$

defines a metric for $\odot$; the corresponding (metric) topology being weaker or equal to the relative topology of $\odot$ as a subset of $w\overline{wB}$.

Proof—It is easily verified that $d(f,g)$ is indeed a metric on $\odot$; lemma 2.4 being used to show that $d(f,g) = 0 \Leftrightarrow f = g$. The second assertion of the lemma follows from the fact that each open ball

$$\left\{f : f \in \odot, \sum_{j=1}^{\infty} \frac{|(f-f_0)\,\dot{X}_j|}{2^j\,(1 + |(f-f_0)\,\dot{X}_j|)} < \delta\right\}$$

of centre f_0 and radius δ contains a neighbourhood

$$V(f_0, \{\dot{X}_j\}_{j=1,2,\ldots,n_\delta}, \delta/2) = \{f : f \in \odot,\ |f(\dot{X}_j) - f_0(\dot{X}_j)| < \delta/2 \text{ for all } X_1, \ldots, X_{n_\delta}\}$$

of f_0 in the relative topology of $\odot$, where n_δ is determined by the condition $1/(2^{n_\delta} - 1) < \delta/2$.

We may now assert the

Theorem 2.6—The unity ball $\odot$, endowed with its relative topology as a subset of $w\overline{wB}$, is a compact metrisable topological space (which henceforward will be denoted by $t\odot$).

Proof—That $\odot$ is a compact subset of $w\overline{wB}$ was established in theorem 2.3. But a Hausdorff topology over a set, weaker than a compact topology over the same set, is necessarily equal to the compact topology.‡ Thus, the metric topology defined on $\odot$ by lemma 2.5 is indeed equal to the relative topology of $\odot$ as a subset of $w\overline{wB}$.

† This may be shown as follows. Let Δ be a closed set of complex numbers. It is easily verified that the weak topology of B is equal to the relative topology of B as a subset of wA. From the continuity of the functionals $X \to \langle Xv, v\rangle$ from wA into $\mathbb{C}$, it follows also that $B = \bigcap_{v \in H} \{X : \langle Xv, v\rangle \in R\}$ is closed in wA. By the continuity of f in wB, we have that the sets $\mathscr{X}_1 = \{X_1 : f(X_1)$ is the real part of some $\lambda \in \Delta\}$ and $\mathscr{X}_2 = \{X_2 : f(X_2)$ is the imaginary part of some $\lambda \in \Delta\}$ are closed in wB and thus closed in wA. Let X^* be the adjoint transformation of $X \in wA$. But then, by the continuity of the mapping $X \to X^*$ in wA (see Dixmier (1950), p. 406), we deduce that the sets $\mathscr{Y}_1 = \{Y : \frac{1}{2}(Y + Y^*) \in X_1\}$ and $\mathscr{Y}_2 = \{Y : (i/2)(Y^* - Y) \in \mathscr{X}_2\}$ are closed in wT. We may conclude, thus, that $f^{-1}(\Delta) = \mathscr{Y}_1 \cap \mathscr{Y}_2$ is a closed subset of wA.

‡ See, for instance, Dunford & Schwartz (1958), i. 5.8.

We intend to use the above theorem to establish that S, endowed with its relative topology as a subset of $\overline{wwB}$, is a metrisable compact topological space. But, in order to do this, we need the following preliminary results:

Lemma 2.7—Let D be the set of all orthogonal projections on the closed linear subspaces of H. Then, if $\{p_j\} \subset S$ is a sequence converging weakly to $f \in \overline{wwB}$, the restriction f/D of f to D coincides with the restriction of some $p \in S$ to the same set.

Proof—For the case in which $\dim H \leqq 2$, we can easily construct directly the p whose existence is asserted. Thus, we may assume that $\dim H \geqq 3$. Let $p_j/_D$ be the restriction of p_j to D, X^0 the projection on the null-subspace of H, X^1 the identity operator on H. Each $p_j/_D$ defines a measure on the set of closed manifolds of H, in the sense of Gleason (1957). Obviously, $f/_D(X^0) = 0$ and $f/_D(X^1) = 1$, with $0 \leqq f/_D(X) \leqq 1$ for any $X \in D$. To show that $f/_D$ also defines a Gleason measure, it must be proved then that $f/_D$ is countably additive. The proof that follows can be found in Gudder (1965), theorem 7.6—it is displayed here for the sake of completeness. Let $\{X_j\}$ be a disjoint sequence of elements of D† and $\mathscr{X} \subset D$ the smallest Boolean σ-algebra of projections containing $\{X_j\}$. By the Loomis representation theorem (see, for instance, Varadarajan (1968), theorem 1.3), there exists a set M, a σ-algebra $\mathscr{M}$ of subsets of M, and a σ-homomorphism h from $\mathscr{M}$ onto $\mathscr{X}$. We have that each p_j induces a measure $\bar{p}_j$ on $\mathscr{M}$ defined by $\bar{p}_j(m) = p_j/_D(h(m))$, $\forall m \in \mathscr{M}$. Defining $\bar{f}(m) = f/_D(h(m))$ we have $\lim \bar{p}_j(m) = \lim p_j/_D(h(m)) = f/_D(h(m)) = \bar{f}(m)$, $\forall m \in \mathscr{M}$. Thus, $\{\bar{p}_j\}$ is a sequence of countably additive scalar functions on the σ-field $\mathscr{M}$, such that $\lim \bar{p}_j(m) = \bar{f}(m)$ for all $m \in \mathscr{M}$. By a theorem of Nikodýn (see Dunford & Schwartz (1958), theorem iii. 7.4), it follows then that $\bar{f}$ is countably additive on $\mathscr{M}$. Let us write $h(m_j) = X_j$ and $t_1 = m_1$, $t_2 = m_2 - m_1$, $t_3 = m_3 - (m_2 \cup m_1)$, ...; obviously $t_k \cap t_\ell = \phi$ for $k \neq \ell$ and $h(t_j) = X_j$ for all j. Hence we have

$$f/_D \sum_j X_j) = f/_D\left[\sum_j h(t_j)\right] = f/_D\left[h\left(\bigcup_j t_j\right)\right] = \bar{f}\left(\bigcup_j t_j\right) = \sum_j \bar{f}(t_j) = \sum_j f/_D h(t_j)$$
$$= \sum_j f/_D(X_j)$$

We conclude, thus, that $f/_D$ is a countably additive non-negative real valued function on D—or, equivalently, on the set of all closed linear manifolds of H. By Gleason's theorem (loc. cit.), we may then assert that $f/_D$ is the restriction to D of some functional belonging to S.

Lemma 2.8—Let $f \in \overline{wwB}$ be a functional such that there exists $p \in S$ for which $f/_D = p$. Then, $f = p$.

Proof (cf. Gudder (1965), theorem 1.4)—Given any $X \in B$ and any $\delta > 0$, we know that there exists a linear combination $\sum_{j=1}^{n} \xi_j X_j$ of elements

† That is to say, $X_{j'} . X_{j''} = X^0$ for $X_{j'}, X_{j''} \in \{X_j\}$ and $j' \neq j''$.

$X_j \in D$, with real coefficients ξ_j, such that $\|X - \sum_{j=1}^n \xi_j X_j\| \leqq \delta$.† Thus, as for any $X \in B$ we have

$$|p(X) - f(X)| \leqq \left|p(X) - p\left(\sum_{j=1}^n \xi_j X_j\right)\right| + \left|p\left(\sum_{j=1}^n \xi_j X_j\right) - f\left(\sum_{j=1}^n \xi_j X_j\right)\right|$$
$$+ \left|f\left(\sum_{j=1}^n \xi_j X_j\right) - f(X)\right| \leqq \left\|X - \sum_{j=1}^n \xi_j X_j\right\| + \|f\| \cdot \left\|\sum_{j=1}^n \xi_j X_j - X\right\|$$

we see that $|p(X) - f(X)|$ is smaller than any arbitrary positive number—that is to say, $f(X) = p(X)$ for any $X \in B$.

We are now ready to prove the

Theorem 2.9—S, endowed with its relative topology as a subset of $w\bar{w}B$, is a metrisable compact topological space.

Proof—It is easily verified that $S \subset \odot$; thus, by showing that S is closed in $t\odot$ such a result will follow immediately from theorem 2.6. Let us show then that if $\{p_j\} \subset S$ is a sequence converging (weakly) to $f \in \odot$, then $f = p$ for some $p \in S$.‡ But this is an immediate consequence of lemmas 2.7 and 2.8.

Corollary 2.10—Any sequence $\{p_j\} \subset S$ has a weakly convergent subsequence—i.e. a subsequence $\{p_{j'}\}$ such that there exists a $p \in S$ for which $p_{j'}(X) \to p(X)$, $\forall X \in B$.

Proof—The corollary follows immediately from theorem 2.9, by standard topological results (see, for instance, Dunford & Schwartz (1958), theorem 1.6.13).

We proceed now to apply corollary 2.10 to the investigation of some 'weak uncertainty principles' which may arise in the context of Quantum Theory.

3. *Uncertainty Bounds: The Sum and the Product of Variances. The Case of Linear Momentum and Position*

In all that follows we assume Schroedinger's formulation of Quantum Mechanics, as usually given in terms of Hilbert-space theory. According to it, pure states and observables correspond, respectively, to unit vectors and self-adjoint operators on the complex separable Hilbert space H of 'wave functions' of the physical system. The pure and the mixed states of

† See, for instance, Riesz & Sz. Nagy (1955), Section 107.

‡ As $w\bar{w}B$ is a Hausdorff topological space and $\odot$ is compact, it follows that $\odot$ is a closed subset of $w\bar{w}B$ and that a sequence $\{p_j\} \subset \odot$, if it converges in the topologies of $w\bar{w}B$ or $t\odot$, may converge only to an element of $\odot$. (Of course, for a sequence of elements of $\odot$, convergence according to the topology of $t\odot$ is equivalent to convergence according to the topology of $w\bar{w}B$.)

the physical system (with phase-space H) are exactly the functionals p of the set S introduced in Section 2.

Let us say that an observable X is *exactly measured in the state* p if its *variance* or *dispersion in* p—i.e. the quantity $\sigma_X^p = p(X^2) - (p(X))^2$—is null. It is easily verified that an observable may be exactly measured only in pure states. Moreover, since von Neumann's studies on the subject (see von Neumann (1932), iii.3) it is well known that X is exactly measured in the pure state $p = \langle \cdot v, v \rangle$ if and only if $v \in H$ is a (normalised) eigenvector of X.

Now, let us suppose X and Y two bounded self-adjoint operators on H. We say that X and Y are *strongly incompatible*, and write $X \leftrightarrow\!\!\!\!\!\!\;+\; Y$, if there is no element in H which is an eigenvector of both X and Y.

From our previous remarks, we have that $X \leftrightarrow\!\!\!\!\!\!\;+\; Y$ implies that there is no state in which both X and Y can be exactly measured. But nothing seems to prevent us from finding states in S under which the variances of both X and Y could be as small as we please. That the situation is not so is the outcome of

Theorem 3.1—Let X and Y be bounded observables (i.e. bounded self-adjoint operators) on the complex separable Hilbert space H. Then, if $X \leftrightarrow\!\!\!\!\!\!\;+\; Y$ there exists a positive real constant η_{XY} such that $\sigma_X^p + \sigma_Y^p \geqq \eta_{XY}$ for any state $p \in S$.

Proof—As noted above, we have necessarily $\sigma_X^p + \sigma_Y^p > 0$ for any state p. Suppose by absurd that no η_{XY} as described does exist. We would have then a sequence $\{p_j\}$ of states for which $\lim_{j\to\infty} (\sigma_X^{p_j} + \sigma_Y^{p_j}) = 0$; by corollary 2.10 we also would have a subsequence $\{p_{j'}\}$ of $\{p_j\}$ converging weakly to some $p_0 \in S$. Now, from $\lim_{j\to\infty} (\sigma_X^{p_j} + \sigma_Y^{p_j}) = 0$ we get $0 = \lim_{j\to\infty} \sigma_X^{p_j}$ and consequently

$$0 = \lim_{j\to\infty} [p_j(X^2) - (p_j(X))^2] = \lim_{j'\to\infty} [p_{j'}(X^2) - (p_{j'}(X))^2]$$
$$= \lim_{j'\to\infty} p_{j'}(X^2) - (\lim_{j'\to\infty} p_{j'}(X))^2 = p_0(X^2) - (p_0(X))^2 = \sigma_X^{p_0}$$

For Y we get analogously $\sigma_Y^{p_0} = 0$. But $\sigma_X^{p_0} = \sigma_Y^{p_0} = 0$ is in contradiction with the fact that $\sigma_X^p + \sigma_X^p > 0$ for any $p \in S$; hence, a η_{XY} as described must exist.

As an illustration we proceed to examine what the above result may tell us about a quite simple physical system—namely, about a quantum-mechanical particle in one dimension, with momentum P and position Q. According to a well-known procedure (see, for instance, Messiah (1965), ch. V, 11), first of all we substitute the observables P and Q (unbounded and with continuous spectra) by their approximate realisations $P_{\xi\xi}$ and $Q_{\xi\xi}$ (bounded and with discrete spectra), constructed as follows.

We can take P and Q as operators acting on adequate linear manifolds of $\mathscr{L}^2(R) \equiv$ the Hilbert space constructed, in the usual way, from the set of all square (Lebesgue) integrable functions $t: R \to \mathcal{C}$.† Let $\bar{\xi}$ and $\mathring{\xi}$ be two positive real numbers such that: (i) the ratio $\bar{\xi}/\mathring{\xi}$ is an even number k; (ii) the position and the momentum coordinates which lie outside the intervals $[-\bar{\xi}/2, \bar{\xi}/2]$ and $[-\pi\hbar/\mathring{\xi}, \pi\hbar/\mathring{\xi}]$, respectively, are beyond the reach of experimental measurability; (iii) a fraction of position and momentum coordinate amounting to $\mathring{\xi}$ and $\pi\hbar/\bar{\xi}$, respectively, is below the level of accuracy of experimental measurement.

It is easy to verify that the family of functions (where $j = 0, \pm 1, \pm 2, \pm 3, \ldots$)

$$v_j(\xi) = \begin{cases} \mathring{\xi}^{-1/2} & \text{if } |j\mathring{\xi}| \leqq |\xi| < |(j+1)\mathring{\xi}| \\ 0 & \text{otherwise} \end{cases}$$

is a set of orthonormal elements of $\mathscr{L}^2(R)$. Then, we take as 'approximate realisation' of the phase space of the system the $(k+1)$-dimensional linear manifold $H_{\bar{\xi}\mathring{\xi}} \subset \mathscr{L}^2(R)$ generated by the functions $v_i(\xi)$ with $j = 0, \pm 1, \pm 2, \ldots, \pm[(k/2) - 1], \pm k/2$. Now we define the 'approximate realisations' $P_{\bar{\xi}\mathring{\xi}}$ and $Q_{\bar{\xi}\mathring{\xi}}$ of P and Q as the operators acting on $H_{\bar{\xi}\mathring{\xi}}$ such that: (i) the eigenvectors of $P_{\bar{\xi}\mathring{\xi}}$ are the functions

$$u_j(\xi) = \begin{cases} \bar{\xi}^{-1/2} \exp(i 2\mathring{\xi} j I(\xi)/\bar{\xi}) & \text{if } \xi \in [-\bar{\xi}/2, \bar{\xi}/2] \\ 0 & \text{otherwise} \end{cases}$$

from R into $\mathcal{C}$, where $j = 0, \pm 1, \pm 2, \ldots, \pm k/2$ and $I(\xi) =$ the largest integer $\leqq \xi/\mathring{\xi}$, the corresponding eigenvalues being given by $\lambda_j = (2\pi\hbar/\bar{\xi}) . j$; (ii) the eigenvectors of $Q_{\bar{\xi}\mathring{\xi}}$ are the functions $v_j(\xi)$ for $j = 0, \pm 1, \ldots, \pm k/2$, the corresponding eigenvalues being $\xi_j = j\mathring{\xi}$.

From the above definitions we have immediately that $P_{\bar{\xi}\mathring{\xi}}$ and $Q_{\bar{\xi}\mathring{\xi}}$, as well as P and Q, are strongly incompatible. Hence, instead of studying the unbounded operators P and Q, we choose to consider their discrete and bounded (although eventually with large norm) approximations $P_{\bar{\xi}\mathring{\xi}}$ and $Q_{\bar{\xi}\mathring{\xi}}$. To this last pair of operators we may apply theorem 3.1, contrariwise to what happens with the original pair, to which the theorem does not necessarily apply, due to the unboundedness of P and Q.

Let us take $\bar{\xi}$ so large and $\mathring{\xi}$ so small that Heisenberg's uncertainty principle applies to $P_{\bar{\xi}\mathring{\xi}}$ and $Q_{\bar{\xi}\mathring{\xi}}$ in the domain of experimental verifiability, without appreciable deviations. With unities adequately normalised, we may then state the usual uncertainty relation in the form $\sigma^p_{P_{\bar{\xi}\mathring{\xi}}} \sigma^p_{Q_{\bar{\xi}\mathring{\xi}}} \geqq 1$, for all p in the set of states of the system with phase-space $H_{\bar{\xi}\mathring{\xi}}$.

We plot the variances $\sigma^p_{P_{\bar{\xi}\mathring{\xi}}}$ and $\sigma^p_{Q_{\bar{\xi}\mathring{\xi}}}$ in a Cartesian graphic (Fig. 1). Each point in the $\sigma^p_{P_{\bar{\xi}\mathring{\xi}}}, \sigma^p_{Q_{\bar{\xi}\mathring{\xi}}}$-plane corresponds then to some set of states of the particle: under each state belonging to such a set, the measurement of P and Q would display the assigned variances.

According to Heisenberg's principle, the physically admissible states belong to the sets represented by points on or above the hyperbola $\sigma^p_{P_{\bar{\xi}\mathring{\xi}}} . \sigma^p_{Q_{\bar{\xi}\mathring{\xi}}} = 1$. On the other hand, theorem 3.1 asserts that the sets of

† As usual, we will speak of the elements of $\mathscr{L}^2(R)$ as functions $t: R \to \mathcal{C}$, although strictly speaking they are actually classes of such functions.

physically admissible states are represented by points on or above the straight line $\sigma^p_{P_{\xi\xi}} + \sigma^p_{Q_{\xi\xi}} = \eta_{P_{\xi\xi}Q_{\xi\xi}}$. If $\eta_{P_{\xi\xi}Q_{\xi\xi}} \leqq 2$, this can be taken merely as a prediction weaker than Heisenberg's.

An estimate of the difference between the result of theorem 3.1 and Heisenberg's can be given by the difference of the areas covered by the points which, in each case, might stand for sets of admissible states. A simple calculation (cf. the Appendix) shows that the 'minimum deviation' between theorem 3.1 and Heisenberg's relation is obtained when $\eta_{P_{\xi\xi}Q_{\xi\xi}} = 2{\cdot}31$. This corresponds to a situation under which states which might be admissible according to Heisenberg would not be so according to theorem 3.1 (and vice versa; see Fig. 1).†

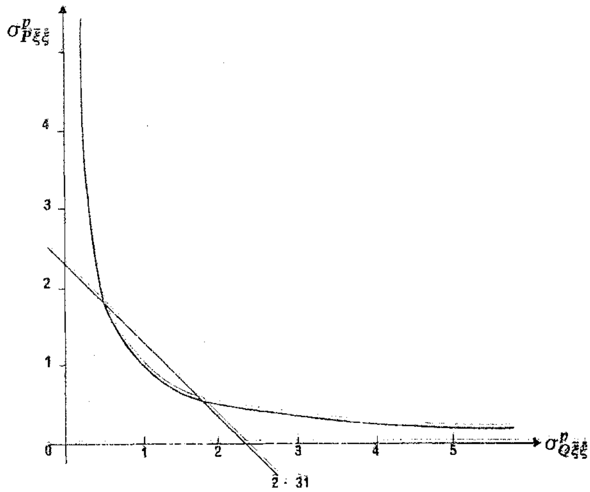

Figure 1

We proceed now to apply theorem 3.1 to the operators corresponding to angular variables of a physical system. But, as the last remark in this section, let us only note that the proof of that theorem can be made much simpler when H is finite-dimensional, as in the example $H = H_{\xi\xi}$ here discussed.

It will be the object of future investigation to try to relate directly the dimensionality of the space $H_{\xi\xi}$ to $\eta_{Q_{\xi\xi}P_{\xi\xi}}$.

† In this paper we will not comment on the actual distribution of sets of (admissible) states in the $\sigma^p_{P_{\xi\xi}}, \sigma^p_{Q_{\xi\xi}}$-plane. The study of such a distribution is an important problem, to be approached in any attempt to check experimentally our results—it seems however to involve rather tedious calculations. We might assume, as a working hypothesis, that the distribution of sets of states according to the variances of $P_{\xi\xi}$ and $Q_{\xi\xi}$ are 'roughly uniform', in a sense that the interested reader will find easy to render precise (in the experiment suggested at Section 5 for instance, it is clear that indeed we have a 'uniform distribution' of the sets of states according to the variances of σ_1 and σ_2).

4. *The Case of Angular Variables*

Let us consider a quantum particle moving along a circumference, with angular momentum $\vec{L}$ and position θ_s. In Schroedinger's formulation of Quantum Mechanics, $\vec{L}$ and θ_s are usually written as the operators

$$L[t(\xi)] = -i\hbar \frac{d}{d\xi}[t(\xi)] \qquad \text{and} \qquad \theta_s[t(\xi)] = \xi[t(\xi)]$$

acting on adequate linear manifolds of $\mathscr{L}^2(R)$. As the functions t belonging to such manifolds must be periodic, with period 2π, it follows then that there is no physical interpretation for the operator $[L, \theta_s]$.

However, it has been shown (cf. Judge, 1963; Judge & Lewis, 1963; Judge, 1964; and Bouten *et al.*, 1965) that actually the observable *position* ought to be represented by the more adequate operator $\theta[t(\xi)] = Y(\xi)[t(\xi)]$, where $Y(\xi) = \xi(\text{mod}.2\pi)$ is a function taking values in $[-\pi, \pi]$. We have thus that

$$[L, \theta] = -i\hbar\left\{1 - 2\pi \sum_{-\infty}^{\infty} \delta[\xi - (2n+1)\pi]\right\}$$

By means of Schwartz's inequality, from the above comutation rule we can prove that

$$\frac{\sigma_L^p \sigma_\theta^p}{\left(1 - \frac{3}{\pi^2}\sigma_\theta^p\right)^2} \geqq \left(\frac{\hbar}{2}\right)^2$$

for any state p of the system; see the last reference just given. Numerical calculations have been carried out with this uncertainty relation, which was found to exhibit appreciable deviations from Heisenberg's (Schotsmans & van Leuven, 1965). Indeed, it is plain that θ being limited, $\sigma_L^p \sigma_\theta^p \geqq (\hbar/2)^2$ is false (consider for instance the case when p is an eigenstate of L!).

We can bring our argument closer to the actual experimental situations (and, at the same time, simplify the above considerations) by restricting the range of ξ to $[-\pi, \pi]$. This implies taking $\mathscr{L}^2([-\pi, \pi])$ as the space containing the wave functions of the system. Discrete and bounded 'approximate realisations' of L and θ can then be constructed, in a way which is formally equivalent to the case of linear momentum and position (see the previous section, putting $\bar{\xi} = 2\pi$ and $\bar{\bar{\xi}} =$ an adequate segment of the orbit of the particle). The same method applied in Section 3 can be used, now, for comparing Judge's uncertainty relation with theorem 3.1. With the system of unities employed in Section 3, we have thus to examine the inequalities

$$\frac{\sigma_L{}^p \sigma_\theta{}^p}{\left(1 - \frac{3}{\pi^2}\sigma_\theta{}^p\right)^2} \geqq 1 \tag{4.1}$$

$$\sigma_L{}^p + \sigma_\theta{}^p \geqq \eta_{L\theta} \tag{4.2}$$

The situation depicted in Fig. 2 must be compared with the one given in Fig. 1. The straight line $\sigma_L^p + \sigma_\theta^p = \eta_{L\theta}$ which is tangent to the hyperbola we now consider—given by $\sigma_L^p \sigma_\theta^p = (1 - (3/\pi^2)\sigma_\theta^p)^2$—is obtained for

$$\eta_{L\theta} = 2\left\{\sqrt{\left[\left(\frac{3}{\pi^2}\right)^2 + 1\right]} - \frac{3}{\pi^2}\right\} \approx 1{\cdot}48$$

According to the calculations indicated in the Appendix, the 'minimum deviation' between inequalities (4.1) and (4.2), in the sense of Section 3, is obtained for $\eta_{L\theta} = 1{\cdot}69$.

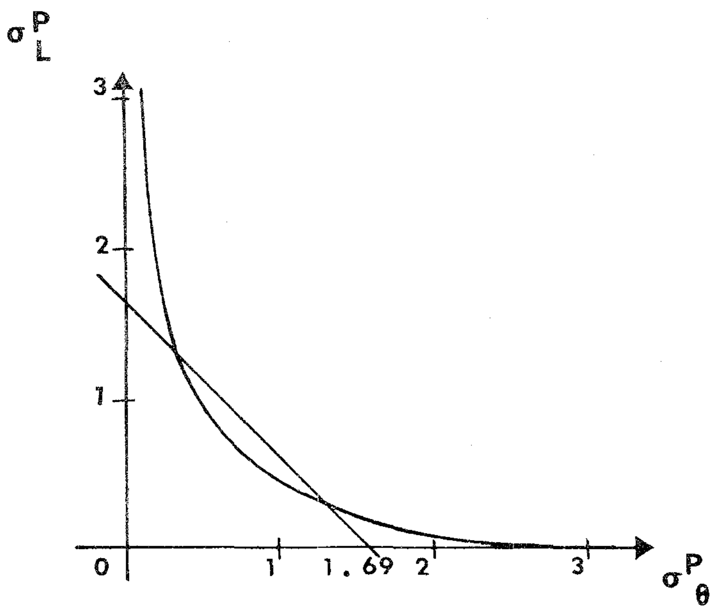

Figure 2

Supposing 1·69 to be the value of $\eta_{L\theta}$, we conclude that states which might be admissible according to Judge's inequality (4.1) would not be so according to inequality (4.2) (and vice versa, see Fig. 2). This situation for angular variables is analogous to the situation for linear momentum and position, examined in Section 3.

From the standpoint of orthodox Quantum Mechanics, the existence of a lower bound for the sum of variances could thus imply in effects similar to the effects brought about by the existence of a superselection rule (see Jauch, 1968; Mackey, 1963). That is to say, the elements of any set of states represented by a point lying between the straight lines $\sigma_X^p + \sigma_Y^p = n_{XY}$ and the corresponding hyperbolas would not be physically realisable if $X = P$, $Y = Q$ and $\eta_{PQ} > 2$ (the case studied in Section 3, cf. Fig. 1) or if $X = L$, $Y = \theta$ and $\eta_{L\theta} > 1{\cdot}48$ (the case examined in this section, cf. Fig. 2).

It would be worthwhile to submit the above conclusion to an experimental test. This could be done by devising an experiment to make obvious the impossibility of realising states which would be forbidden under the present theory. We have a suggestion to put forward, concerning such an experimental arrangement, and it refers to particles with spin.

Let us emphasise that our experiment, as described in the next section, is not merely a 'Gedanken experiment'—for it *can* and we hope it *will* be eventually carried out.

We start by examining the general case of angular momentum and then proceed to the case of the spin 1/2.

5. *Uncertainty Relations and the Angular Momentum. The Particle with Spin* 1/2

Let $\vec{L}$ be the angular momentum of some physical system. By L_1, L_2 and L_3 we will denote the self-adjoint linear operators† which stand for the components of $\vec{L}$ according to some Cartesian frame. L^2 will be the self-adjoint operator† representing the square $\vec{L}.\vec{L}$ of the angular momentum. Then, from the orthodox formalism of Quantum Mechanics we have that the following commutation rules must hold (in this section the unities are chosen such that $\hbar = 1$).

$$[L_h, L_j] = iL_k \qquad \text{for } h, j, k = 1, 2, 3 \text{ and } h \neq j, h \neq k, j \neq k$$
$$[L^2, L_j] = 0 \qquad \text{for } j = 1, 2, 3$$

We have besides that the eigenvalues of L^2 are $\ell = n(n/2 + 1)$, where n runs over the non-negative integers. For the value ℓ of $\vec{L}.\vec{L}$, any one of the components of $\vec{L}$ has $-\ell, -\ell + 1, \ldots, \ell - 1, \ell$ as its admissible values. Actually, if $\vec{L}.\vec{L} =$ some constant value ℓ_0, the operators L_1, L_2 and L_3 can be considered as acting on a $(2\ell_0 + 1)$ dimensional subspace of $\mathscr{L}^2(R)$.

Finally, supposing $\ell \neq 0$ we also know from the orthodox formalism that L_j and L_k have no common eigenvector if $j \neq k$.

Let us assume then that our physical system has a constant non-null square angular momentum $\vec{L}.\vec{L}$. Then, the components L_1, L_2, L_3 are (bounded) self-adjoint linear operators on a finite-dimensional Hilbert space. Furthermore, we can apply to such components the theorem 3.1: for all states p of the system and $j \neq k$ we have $\sigma^p_{L_j} + \sigma^p_{L_k} \geqq \eta_{L_j L_k} > 0$.

If we try to get an 'uncertainty relation' on the product $\sigma^p_{L_j}\sigma^p_{L_k}$ by the orthodox method (see, for instance, Jauch (1968), 11.1), we obtain however nothing more besides the trivial inequality $\sigma^p_{L_j} . \sigma^p_{L_k} \geqq 0$. Thus, we have here a case in which the 'weak uncertainty principle' WUP of the Introduction brings forth some interesting new information, not derivable from the orthodox indeterminacy principles.

† Acting on adequate linear manifolds of $\mathscr{L}^2(R)$.

It would be worthwhile to examine physical situations which, despite obeying the condition of WUP, under a first scrutiny would seem able to exhibit states violating some specific form of WUP. We proceed thus to analyse an experiment with spin, although the orthodox formalism of Quantum Mechanics tells us from the start that WUP must be obeyed then with 1 as the infimum for the sum of variances. The experiment would allow us to locate—by direct empirical verification—the value of $\eta_{L_j L_k}$. We might even have the effect described at the end of Section 4, i.e. the ruling out of certain states which would be actually realisable according to orthodox quantum theory.

In order to explain our experiment, let us start with some general considerations. The object of the experiment would consist of a stream of particles with spin 1/2, in translation along some axis. The components of the spin $\vec{S}$ of any of such particles can be represented by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We note that the operators σ_1, σ_2, σ_3† obey the same commutation rules satisfied by the components of angular momentum and that $\vec{S}.\vec{S}$ = constant $\frac{3}{4}$. Thus, all the previous considerations about L_1, L_2, L_3 and $\sigma^p_{L_1}$, $\sigma^p_{L_2}$, $\sigma^p_{L_3}$ apply also to σ_1, σ_2, σ_3 and $\sigma^p_{\sigma_1}$, $\sigma^p_{\sigma_2}$, $\sigma^p_{\sigma_3}$.‡

In what refers to the spin, a (pure) state p of a particle in the stream can be defined by a unit vector v_p in the space $\mathbb{C}^2$. Let us take the basis in $\mathbb{C}^2$ given by the eigenvectors (1,0) and (0,1) of σ_3. Physically, this means taking a preferred direction in space, defined by σ_3 and to be determined by an adequate magnetic field.

Suppose now that the (pure spin) state of the particles is represented by the vector $v_p = c_\uparrow(1,0) + c_\downarrow(1,0)$ where $c_\uparrow$ and $c_\downarrow$ are complex coefficients such that $|c_\uparrow|^2 + |c_\downarrow|^2 = 1$. Then:

(A)—From the definition of variance we have $\sigma^p_{\sigma_1} = 1 - (2\,\text{Real}\,\bar{c}_\uparrow c_\downarrow)^2$ and $\sigma^p_{\sigma_2} = 1 - (2\,\text{Real}\,i.\bar{c}_\uparrow c_\downarrow)^2$ for any (pure spin) state p of the particles. Furthermore, from the condition $|c_\uparrow|^2 + |c_\downarrow|^2 = 1$ it follows that $|c_\uparrow . c_\downarrow| \leqslant \frac{1}{2}$ and thus $\sigma^p_{\sigma_1} + \sigma^p_{\sigma_2} \geqslant 1$.¶

(B)—From theorem 3.1, we have $\sigma^p_{\sigma_1} + \sigma^p_{\sigma_2} \geqq \eta_{\sigma_1 \sigma_2}$ for some $\eta_{\sigma_1 \sigma_2} \geqq 0$ and any (pure spin) state p of the particles.

From (A) and (B) some interesting facts do follow. First of all, (A) establishes a 'weak uncertainty relation' between σ_1 and σ_2, which is usually overlooked in the current discussions about spin 1/2. Second, we see that it makes sense to ask, what is the value of the constant $\eta_{\sigma_1 \sigma_2}$, mentioned

† Acting on the complex Hilbert space $\mathbb{C}^2$.

‡ Cf. Messiah (1965), ch. XIII, I.

¶ For

$$\begin{aligned} \sigma^p_{\sigma_1} + \sigma^p_{\sigma_2} &= 2 - 4[(\text{Real}\,\bar{c}_\uparrow c_\downarrow)^2 + (\text{Real}\,i\bar{c}_\uparrow c_\downarrow)^2] \\ &= 2 - 4[(\text{Real}\,\bar{c}_\uparrow c_\downarrow)^2 + (\text{Imag}\,\bar{c}_\uparrow c_\downarrow)^2 \\ &= 2 - 4|c_\uparrow . c_\downarrow|^2 \geqq 2 - 1 = 1 \end{aligned}$$

in (B)? If it happens that $\eta_{\sigma_1\sigma_2} \leqq 1$, (B) would tell us nothing more than it is already said in (A). However, if it happens that $\eta_{\sigma_1\sigma_2} > 1$, new information would be brought about by (B). We are now ready to describe the experiment to determine which of these two alternatives does really occur.

We consider a Stern–Gerlach apparatus SG_1 which would split an unpolarised stream of neutrons—travelling along the $\vec{y}$-axis of an orthogonal frame $\vec{x}$, $\vec{y}$, $\vec{z}$—in two beams located in the $\vec{y}$, $\vec{z}$-plane. The operators σ_1, σ_2 and σ_3 are supposed to give the spin components according to the $\vec{x}$, $\vec{y}$ and $\vec{z}$, respectively.

Let SG_0 be another Stern–Gerlach apparatus, similar to SG_1 but suitably modified in order to send a stream of (completely) polarised neutrons along $\vec{y}$. Such a beam of particles would then be fed into SG_1. We assume, furthermore, that the spin magnetic momentum of the polarised neutrons would be oriented according to (the positive sense of) some $\vec{z}^0$-axis orthogonal to $\vec{y}$.

Denoting by (1,0), (0,1) the eigenvectors of σ_3 and by α the angle between $\vec{z}$ and $\vec{z}^0$,† we have thus that our particles would enter SG_1 in the (pure spin) state

$$v_{p(\alpha)} = \cos\frac{\alpha}{2}(1,0) + \operatorname{sen}\frac{\alpha}{2}(0,1)$$

As we would have

$$\sigma_{\sigma_1}^{p(\alpha)} + \sigma_{\sigma_2}^{p(\alpha)} = 2 - \operatorname{sen}^2\alpha$$

it follows that if $\eta_{\sigma_1\sigma_2} > 1$, then some states $v_{p(\alpha)}$ with $\alpha \approx \pi/2$ would be forbidden by (B), despite being physically realisable according to the orthodox context of Quantum Mechanics.

We believe that the occurrence of forbidden states may reveal itself by the absence of output from SG_1, when $\vec{z}^0$ is tilted by $\pi/2$ around $\vec{y}$ or, still, it may be revealed by disturbances in the intensities of the beams emerging from SG_1, when $\alpha \approx \pi/2$.

About the experimental conditions, we must take into account the interference of thermic fluctuations in the measurements. The energy of interaction spin-magnetic field for each neutron must be significantly greater than the thermic energy kT (here k stands for Boltzmann's constant). As the magnetic momentum of the neutron amounts to 10^{-23} erg/oersted, given an external magnetic field of 10^4 oersted we have around 10^{-19} ergs for the energy of interaction spin-external field. Thus, we would suggest: (a) taking into SG_1 thermal neutrons (for example, with energy around 10^{-2} eV); (b) maintaining an adequate part of the experimental arrangement under a temperature significantly below 10^{-3} °K.

Finally, let Δt be the time of transit of the particles between SG_0 and SG_1. Considering the question of the collapse of the neutrons into a polarised

† α is supposed to be counted from $\vec{z}$, in the positive sense defined by the system $\vec{x}, \vec{y}, \vec{z}$.

state, we deduce that it would be convenient to make $\Delta t > 10^{-13}$. About the time of transit we note however that if Bohm and Bub's 'hidden variables' theory is valid, then 'unorthodox' results may eventually arise, depending on the value of Δt—cf. Bohm & Bub, 1966, especially Section 7, and Tutsch, 1968.

Acknowledgements

Sections 2 and 3 of the present paper contain an Hilbert-space version of ideas developed, within an abstract frame, in the 1969 Ph.D. dissertation of one of the authors (F. A. J.). Thanks are due to Professor C. W. Kilmister of King's College, University of London, under whose advice the dissertation was written. The financial support of the following entities, towards the realisation of such a previous work, is also gratefully acknowledged: CAPES-Brazilian Ministry of Education and Culture, University of Brasilia and Divisão de Cooperação Intelectual—Brazilian Ministry of Foreign Affairs.

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APPENDIX

Estimate of the 'minimum deviation' between $x.y = 1$ and $x + y = \eta$ (cf. Section 3)

We intend to estimate the area $A(\eta)$ which is given by the shadowed section of Fig. 3 (if $\eta \leqq 2$) or Fig. 4 (if $\eta > 2$).

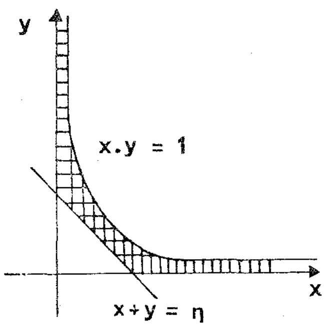

Figure 3

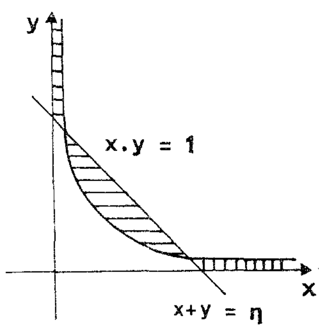

Figure 4

Supposing that x and y take (all the) values in some interval $[0, \bar{\eta}]$,† an elementary calculation gives us, if $0 \leqq \eta \leqq 2$,

$$A(\eta) = 1 - \frac{\eta^2}{2} + 2 \log \bar{\eta}$$

† Here $\bar{\eta}$ is taken sufficiently large.

and if $2 \leqq \eta \leqq \bar{\eta}$,

$$A(\eta) = -\frac{\eta^2}{2} + \eta\sqrt{(\eta^2 - 4)} + 4\log\frac{2\sqrt{\bar{\eta}}}{\eta + \sqrt{(\eta^2 - 4)}} + 1$$

The function $A(\eta)\colon [0, \bar{\eta}] \to R$ defined by the expressions above is obviously continuous and decreasing at least up to $\eta = 2$. It is also easily seen that $A(\eta)$ must have a minimum for $\eta_{\min} \in (2, \bar{\eta}]$. Posing $\eta = 2\cosh\theta$ a good approximation for the increment ΔA of A, when η is varied by $\Delta\eta$, is given by

$$\Delta A = A(\eta + \Delta\eta) - A(\eta) \approx \sqrt{[2(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta})^2]}\frac{\Delta\eta}{\sqrt{2}} - 2\sqrt{[(\eta - \mathrm{e}^{\theta})^2 + \mathrm{e}^{-2\theta}]}\frac{\Delta\eta}{\sqrt{2}}$$
$$= (\mathrm{e}^{\theta} - 3\mathrm{e}^{-\theta})\Delta\eta$$

Making $\Delta\eta \to 0$, we get then $dA/d\eta = \mathrm{e}^{\theta} - 3\mathrm{e}^{-\theta}$ and we may deduce that $A(\eta)$ attains its minimum when $\mathrm{e}^{\theta} = 3\mathrm{e}^{-\theta}$. An approximate solution of this equation is $\theta \approx 0{\cdot}55$, for which $\eta \approx 2{\cdot}31$.

Of course, we do not claim that the last arguments above are mathematically flawless—but it can be easily verified that indeed $\eta \approx 2{\cdot}31$ is a point of minimum for $A(\eta)$. That such minimum is unique seems to be a safe guess....

Estimate of the 'minimum deviation' between $\dfrac{x \cdot y}{\left(1 - \dfrac{3}{\pi^2}x\right)^2} = 1$ *and* $x + y = \eta$ (*cf. Section* 3)

Let us put $3/\pi^2 = \gamma$; the point at which the straight line $x + y = \eta$ is tangent to the hyperbola $xy/(1 - \gamma x)^2 = 1$ is easily determined as $x = 0{\cdot}956$ and $y = 0{\cdot}526$. The value of η for the tangent line is $1{\cdot}482$. As in the previous case, we look for a value of $\eta > 1{\cdot}482$ which would 'minimise the deviations' between the previsions given by our two equations. We can safely assume that such value of η is $\leqq 1/\gamma$.

For $1{\cdot}482 < \eta \leqq 1/\gamma$, the area $A(\eta)$ to be minimised is given by

$$A(\eta) = A_0 + \int_{x_0}^{x_1} \frac{(1 - \gamma x)^2}{x}\,dx - \int_{x_0}^{x_1} (\eta - x)\,dx + \int_{x_1}^{x_2} (\eta - x)\,dx$$
$$- \int_{x_1}^{x_2} \frac{(1 - \gamma x)^2}{x}\,dx + \int_{x_2}^{1/\gamma} \frac{(1 - \gamma x)^2}{x}\,dx - \int_{x_2}^{\eta} (\eta - x)\,dx$$

where $x_0 > 0$ is an arbitrary constant close to 0, introduced to avoid the divergence of the first integral. The constant A_0 depends on x_0 and on the maximum value taken by the variable y (which is supposed bounded, cf. Section 3). The coordinates x_1 and x_2 are the abscissas of the points where the straight line $y = \eta - x$ cuts the hyperbola $y = (1 - x)^2/x$, respectively

$$\frac{(2\gamma + \eta) \mp \sqrt{[(2\gamma + \eta)^2 - 4(\gamma^2 + 1)]}}{2(\gamma^2 + 1)}$$

Substituting the integrals above by their algebraic expressions and deriving the result in relation to η, we find after some simplifications that

$$\frac{dA}{d\eta} = 2{\cdot}807\eta^4 + 8{\cdot}277\eta^3 - 21{\cdot}353\eta^2 - 38{\cdot}912\eta + 63{\cdot}941$$

Two of the roots of this equation are negative and one is $<1{\cdot}482$. The remaining one is the value we search for, 1·69.

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